

Entangled States and Super-radiant Phase Transition

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Abstract

The Dicke spin-boson model is composed by a single bosonic mode and an ensemble of N identical two-level atoms. Assuming thermal equilibrium with a reservoir at temperature β^{-1} , we consider the situation where the coupling between the bosonic mode and the atoms generates resonant and non-resonant processes. The thermodynamic of the model is investigated. Next we introduce dipole-dipole interaction between the atoms. We investigate the transition from fluorescent to super-radiant phase and the quantum phase transition in a situation where the dipole-dipole interaction between the atoms generates entangled states in the atomic system. We proved that, the critical behavior is not modified by the introduction of the dipole-dipole interaction.

PACS numbers:

42.50.Nn, 05.30.Jp, 73.43.Nq

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1 Introduction

According to classical physics, the knowledge of the state of a composed system yields complete knowledge of the individual state of the parts. This is the principle of separability [1] [2]. For example, if we know the state of a classical system of particles, we know the state of each particle. However, this is not true in the context of quantum mechanics. One of the consequences of the superposition principle is the introduction of the concept of entanglement. After interacting, two quantum systems can end up in a non-separable state, i.e., states that can not be factorized into a product of the states of its sub-systems. These states of such composite system are called entangled states [3]. For instance, in a bipartite entangled state, each part loses its quantum identity. Entanglement leads to the non-locality properties of quantum mechanics, and action at a distance at a speed greater than the speed of light [4] [5]. Discussing the hidden variable theories based on local realism, Bell has shown that quantum theory is supposed to be non-local [6] [7] [8]. It was shown that any pure entangled state of two spin- $\frac{1}{2}$ violates corresponding Bell's inequality [9] [10] [11] [12].

With the development of quantum information [13] [14] [15] and its application in computation and communication [16] [17] [18] [19] [20] [21] [22] [23], the entangled states have been attracted enormous interest. For instance, new tests of quantum mechanics can be implemented using some entangled states, as for example the Bell and the GHZ states [24] [25]. Also, several quantum protocols can be realized exclusively with the help of entangled states [26] [27], as for example the quantum dense coding protocol proposed by Bennett and Wiesner [28], or the quantum teleportation protocol proposed by Bennett and co-workers [29]. In the teleportation of entanglement a single qubit in an arbitrary state can be transferred, exchanging information over long distances, without physically transferring the system itself. The maximally entangled pair of qubits is crucial in the quantum teleportation. Actually, there are many physical systems that can be used to implement a quantum computer with the quantum logic gates. Some experimental devices are based on cavity quantum electrodynamics, trapped ions and nuclear magnetic resonances. For example, for the realization of quantum logic gates, a many-body system prepared in an entangled state has been proposed [30]. Another promising system used to implement quantum computation is the quantum dot array proposed by Loss and DiVincenzo [31]. The basic problem that arises in this area of research is how to create entangled in many-body systems [32] [33] [34] and also how to generate systems which are not affected by the environment, overcoming the problem of decoherence.

In the simplest case of two atoms, it is quite important to demonstrate creation of entanglement on such system. There are in the literature different methods for detection of entangled states of two interacting atoms. One method is based in measure the angular intensity distribution of the fluorescent field emitted by these two atoms, since it is well known that radiation emitted by atoms exhibits directional properties [35] [36] [37] [38] [39] [40]. The other is based in the dynamic of the population inversion of the system. For instance, the properties of spontaneous emission from two identical entangled atoms interacting with the modes of a bosonic field was investigated by Guo and Yang [41]. They have shown that the time evolution of the population inversion,

which is proportional to the radiation intensity, depends on the degree of entanglement of the initial state of the system.

Suppose that in a many-body system an entanglement has been created in a given portion of such system. A fundamental question is how to know whether a state of a many-body system is entangled or not [42]. Before continue, we would like to briefly describe the theoretical machinery that are needed to define entanglement in a pure or mixed state. To define separability and entanglement in bipartite systems, we use the properties of the state vector or density operator. Since a pure state is entangled if it is not separable, it can be shown that a pure state is separable if and only if the reduced density operators of sub-systems represent pure states. Therefore a pure state is entangled if and only if the reduced density operators for the sub-systems describe mixed states. In practice, a pure state is separable if and only if the quantum fluctuations of all observable (linear self-adjoint operators acting in the respective Hilbert space of each sub-system) are uncorrelated. If at least one pair of linear self-adjoint operators have correlated quantum fluctuations, the pure state is entangled. For mixed states we define a separable mixed state if the two sub-systems have the same purity and von Neumann entropy. An entangled mixed state is one that is not separable. We would like to stress that experimental tests for separability and entanglement of pure states using correlated quantum fluctuations are not available for mixed states. Strong correlation between two operators does not mean that there is entanglement for a system described by a mixed state.

Critical phenomena is a cooperative effect characterized by fluctuations in the order parameter, where all scales of length are important. For instance, the two-point correlation function (the expectation value of products of local observables) near the phase transition does not decrease exponentially, but polynomially, as the distance between the points goes to infinity. The system exhibit long range order. Since, for pure states the correlated quantum fluctuations of two observables defines an entangled state, we should expect a close connection between second order phase transition at zero temperature and entanglement. We would like to note that recently Emary and Brandes [43] [44] discussed the connection between quantum phase transition and the chaotic behavior that emerges in the full Dicke model for finite N , where the energy level-spacing statistics changes from Poissonian to one described by Gaussian ensembles of the random matrix theory [45] [46].

Let us now focus our attention for the super-radiant phase transition. Consider an ensemble of N two-level atoms, all prepared into the excited state. Each atom can emit a photon by spontaneous emission. In the situation where there is not coupling between the atoms, each atom radiates independently. This is called the fluorescent phase. Let us consider the following experimental environment: the N two-level atoms are in an optical cavity where all atoms are in resonance with a single mode of the field. For an ensemble of atoms in a volume with linear dimensions small compared to the emission wavelength, they start to radiate spontaneously faster and strongly than the ordinary fluorescent phase. In this situation the radiation rates becomes quadratic dependent on the number of atoms. This cooperative process is called super-radiance. Recently Lambert and co-workers [47] investigated the entanglement properties of an ensemble of N two-level atoms interacting with a field mode in the super-radiant phase [48] via the von

Neumann entropy. The maximum entanglement occurs near the critical region. The authors also studied the system at zero temperature and how does the entanglement is affected by a quantum phase transition that occurs in the system [49] [50]. By the other hand, in a system of N atoms, the presence of the dipole-dipole interaction can generates entangled states in the many-body system. See for example Ref. [51], where was discussed in detail how to generate multi-particles entanglement of atoms trapped by harmonic potential, interacting with a classical field and between them via the dipole-dipole interaction. Therefore one can imagine that if entangled has been created in a portion of the many-body system, the critical behavior of the system changes due to the strong correlation between the atoms.

The aim of this work is to investigate if the entanglement between two-level atoms, generated by the dipole-dipole interaction, is able to change the critical properties of the system. For instance, it is well known that the dipole-dipole decohering effect is able to suppress the super-radiant emission [52] [53]. Note that we are not able to quantify the degree of entanglement present in the model since we are using functional integration methods to investigate the thermodynamic of the model. First, we investigate the full Dicke model where the quantum phase transition and a phase transition from fluorescence to super-radiant phase, at some temperature β^{-1} , in the system of N atoms interacting with a bosonic field is analyzed. Second, introducing the dipole-dipole interaction we discuss the effects of entanglement in this many-body system, showing that the critical temperature of the transition from the fluorescent to super-radiant phase is not modified in this case. We proved that the spectrum of the collective bosonic excitations of the model is unaffected by the dipole-dipole interaction. In this case, there is also a quantum phase transition, at some values of the physical parameters of the model. For the reader interested in the the study of entanglement in many-body systems close to the quantum phase transition, see for example the Refs. [54] [55] [56].

Recently, using the path integral approach with functional integration method the analytic behavior of thermodynamic quantities in the full Dicke model [57] [58] [59] [60] [61] [62] was presented [63]. See also the Refs. [64] [65]. The full Dicke model is similar to the Jaynes-Cummings model [66] where the non-resonant processes in which the atom and the field are excited or de-excited simultaneously, known in the literature as the anti-Jaynes-Cummings model. In Ref. [63] the study of the nonanalytic behavior of thermodynamic quantities in the full Dicke model, allow the authors to evaluate the critical transition temperature. It was shown that the system develop a quantum phase transition and a phase transition from ordinary fluorescence to super-radiant phase at some critical temperature. Based in this analysis, the spectrum of the collective bosonic excitations, for different situations were presented. It was study the rotating-wave approximation, the counter rotating-wave approximation and also the general case. As we discussed, the rotating-wave approximation ignores energy non-conserving terms in which the emission (absorption) of a quantum of a quantized field is accompanied by the transition of one atom from its lower (upper) to its upper (lower) state. In a situation where only non-resonant processes contribute, the full Dicke model present a second order phase transition from the ordinary fluorescent to the super-radiant phase respectively, at some critical temperature β_c^{-1} and also a quantum phase transition, i.e., a phase transition at zero temperature. In the last case there are no thermal excitations,

therefore the phase transition is driven by the quantum fluctuations. The interesting result is the fact that it is possible to have a condensate with super-radiance in a system of N two-level atoms coupled with one mode of a bosonic field where only non-resonant processes contribute.

In the Ref. [67], still neglecting the direct interaction between the atoms, it was considered also two different models, assuming that a single quantized mode of a bosonic field interacts with a ensemble of N identical two-level atoms. Again we assume that the system is in thermal equilibrium with a reservoir at temperature β^{-1} . Analytic properties of the partition functions of the models were also investigated. First, it was study a modified version of the model discussed by Chang and Chakravarty, Legget and others [68] [69] [70] [71], which has been used to analyze dissipation in quantum computers. Owing to the coupling between the two-level systems and a bosonic reservoir, this model presents destruction of quantum coherence without decay of population. Since the interaction Hamiltonian of the model generates a quite particular non-resonant processes, it is shown that the partition function is analytic for all temperatures, and therefore there is no second order phase transition in the model. Second, it was investigated a model where the coupling between the bosonic mode and N two-level atoms is intensity dependent [72] [73] [74], introducing also the couplings g_1 and g_2 for rotating and counter-rotating terms respectively. At low temperatures, the contribution coming from the counter-rotating terms dominates over the rotating ones. When the coupling constant is given by $g_2 = (\omega_0 \Omega)^{\frac{1}{2}}$, where ω_0 is the energy of the single-mode bosonic field and Ω is the energy gap of the atoms, a quantum phase transition appears.

In this paper the physical system in consideration is an ensemble of two-level atoms interacting with a single mode of a bosonic field. It is a natural question to ask if non-Gaussian terms can change the critical behavior of this spin-boson model. We study two models, the full Dicke model and the same model where we include the dipole-dipole interaction term between the two-level atoms. We proved that the critical temperature that characterize the phase transition from fluorescent to super-radiant phase is the same in both models. Also the quantum phase transitions, at some values of the physical parameters of both models are the same. We are using the path integral approach with the functional integration method to investigate the thermodynamic of the models, which is given by the analytic properties of the partition function. The paper is organized as follows. In section II we present the fermionic full Dicke model and the model with the dipole-dipole interaction. In section III we discuss the functional integral for the full Dicke model. In section IV we study the full Dicke model with the dipole-dipole interaction. Conclusion are given in section V. In the Appendix the theory of pure and mixed ensembles and the reduced density operator is briefly presented. In the paper we use $k_B = c = \hbar = 1$.

2 The N two-level atoms-Bose field interaction Hamiltonians

In this section we consider a very general situation where the system under investigation contains a large number of two-level atoms. In order to describe the dynamics of the the two-level atoms and the bosonic mode we have to introduce the Hamiltonian governing the interaction of the quantized Bose field with free atoms [75] [76]. Free means that there is no interaction between the atoms. Therefore let us consider a Bose quantum system B , with Hilbert space $\mathcal{H}^{(B)}$ which is coupled with N atoms, with Hilbert space $\mathcal{H}^{(Q)}$. Let us assume that the whole system is in thermal equilibrium at temperature β^{-1} . The Bose quantum system is a sub-system of the total system living in the tensor product space $\mathcal{H}^{(B)} \otimes \mathcal{H}^{(Q)}$. Let us denote by H_B the Hamiltonian of the quantized Bose field, by H_Q the free Hamiltonian of the N two-level atoms and H_I the Hamiltonian describing the interaction between the quantized Bose field and the N two-level atoms. The Hamiltonian for the total system can be written as

$$H = H_B \otimes I_Q + I_B \otimes H_Q + \delta H_I, \quad (1)$$

where I_B and I_Q denotes the identities in the Hilbert spaces of the quantized Bose field and the N atoms. In the Eq. (1), δ is a small coupling constant between the atoms and the quantized Bose field.

The main purpose of this section is to discuss N -atoms-Bose field interaction Hamiltonian. Therefore, let us introduce the Dicke operators to describe each atom. The free j -th atom Hamiltonian will be denoted by $H_D^{(j)}$, since we are using the Dicke representation. Therefore, we have

$$H_D^{(j)} |i\rangle_j = \omega_i^{(j)} |i\rangle_j, \quad (2)$$

where $|i\rangle_j$ are orthogonal energy eigenstates accessible to the j -th atom and $\omega_i^{(j)}$ are the respective eigenfrequencies. Using Eq. (2) and the orthonormality of the energy eigenstates we can write the j -th atom Hamiltonian $H_D^{(j)}$ as

$$H_D^{(j)} = \sum_{i=1}^2 \omega_i^{(j)} (|i\rangle \langle i|)_j. \quad (3)$$

Let us define the pseudo-spin operators $\sigma_{(j)}^z$, $\sigma_{(j)}^+$ and $\sigma_{(j)}^-$ for each atom by

$$\sigma_{(j)}^z = \frac{1}{2} (|2\rangle \langle 2| - |1\rangle \langle 1|)_j, \quad (4)$$

$$\sigma_{(j)}^+ = (|2\rangle \langle 1|)_j, \quad (5)$$

and finally

$$\sigma_{(j)}^- = (|1\rangle\langle 2|)_j. \quad (6)$$

Combining Eq. (3) and Eq. (4), the j -th atom Hamiltonian can be written as

$$H_D^{(j)} = \Omega^{(j)} \sigma_{(j)}^z + \frac{1}{2}(\omega_1^{(j)} + \omega_2^{(j)}), \quad (7)$$

where the energy gap between the energy eigenstates of the j -th atom is given by

$$\Omega^{(j)} = \omega_2^{(j)} - \omega_1^{(j)}. \quad (8)$$

Shifting the zero of energy to $\frac{1}{2}(\omega_1^{(j)} + \omega_2^{(j)})$ for each atom, the j -th atomic Hamiltonian given by Eq. (7) can be rewritten as

$$H_D^{(j)} = \Omega^{(j)} \sigma_{(j)}^z. \quad (9)$$

Note that the pseudo-spin operators $\sigma_{(j)}^+$, $\sigma_{(j)}^-$ and $\sigma_{(j)}^z$ satisfy the standard angular momentum commutation relations corresponding to spin $\frac{1}{2}$ operators, i.e.,

$$[\sigma_{(j)}^+, \sigma_{(j)}^-] = 2\sigma_{(j)}^z, \quad (10)$$

$$[\sigma_{(j)}^z, \sigma_{(j)}^+] = \sigma_{(j)}^+, \quad (11)$$

and finally

$$[\sigma_{(j)}^z, \sigma_{(j)}^-] = -\sigma_{(j)}^-. \quad (12)$$

The model that we are interested is composed by the Hamiltonian of the atoms, with the contribution of the quantized Bose field H_S , and the interaction Hamiltonian $H_I^{(j)}$. The Hamiltonian of the total system is given by

$$H = I_B \otimes H_Q + H_B \otimes I_Q + \delta H_I = I_B \otimes \Omega \sum_{j=1}^{\infty} \sigma_{(j)}^z + \sum_{k=1}^{\infty} \omega_k b_k^\dagger b_k \otimes I_Q + \frac{1}{\sqrt{N}} \sum_{j=1}^N \sum_{k=1}^{\infty} g_{kj} (b_k + b_k^\dagger) \otimes (\sigma_{(j)}^+ + \sigma_{(j)}^-). \quad (13)$$

where the first summation in the right hand side is

$$\sum_{j=1}^N \Omega^{(j)} \sigma_{(j)}^z = \Omega^{(1)} \sigma_{(1)}^z \otimes \mathbf{1} \otimes \dots \otimes \mathbf{1} + \dots + \mathbf{1} \otimes \mathbf{1} \otimes \dots \otimes \mathbf{1} \otimes \Omega^{(N)} \sigma_{(N)}^z, \quad (14)$$

and $\mathbf{1}$ denotes the identity in the Hilbert space of each two-level system. The quantity g_{kj} is the coupling coefficient. Each two-level system is described by a Hilbert space $\mathcal{H}^{(1)}, \mathcal{H}^{(2)}, \dots, \mathcal{H}^{(N)}$. The composite atomic system is described by the N -fold tensor product space $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} \otimes \mathcal{H}^{(3)} \dots$

One point which is important to stress is that in the Eq. (13) there are terms so called counter-rotating terms. In the rotating-wave-approximation we ignore energy non-conserving terms in which the emission (absorption) of a quantum of a quantized field is accompanied by the transition of one atom from its lower (upper) to its upper (lower) state. As we discussed before, the rotating-wave-approximation ignores terms in which the j -th atomic raising (lowering) operators multiplies the field creation (annihilation) operator.

We can also introduce an atom-atom interaction, which is relevant in the study of entangled states. In an entangled system, the state of the composite system can not be factorized in to a product of the states of its sub-systems. For example in the case of two two-level atoms, the interaction Hamiltonian of two atoms including the dipole-dipole interaction reads

$$H_{(qq)} = \frac{1}{N} \sum_{i \neq j}^2 H_{(ij)} \sigma_{(i)}^+ \otimes \sigma_{(j)}^- . \quad (15)$$

Let us note that the interaction Hamiltonian given by Eq. (15) is a particular case of the quantum Hamiltonian describing an axially symmetric ferromagnet, with the coupling constants J_{ij}^z and J_{ij} . In terms of the pseudo spin operators we have

$$H = - \sum_{\langle i,j \rangle} \left(J_{ij}^z \sigma_{(i)}^z \sigma_{(j)}^z + J_{ij} \left(\sigma_{(i)}^+ \sigma_{(j)}^- + \sigma_{(i)}^- \sigma_{(j)}^+ \right) \right) . \quad (16)$$

The $\langle i, j \rangle$ symbol denotes a summation over pair of first neighbors. In the absence of the "dipole-dipole" interaction the pure Hilbert space of the two atomic system is spanned by the states $|g_1\rangle \otimes |g_2\rangle, |g_1\rangle \otimes |e_2\rangle, |e_1\rangle \otimes |g_2\rangle$ and $|e_1\rangle \otimes |e_2\rangle$, where g and e denotes respectively the ground and the excited state of the two atomic system. If we include the dipole-dipole interaction term in the form of Eq. (15), the vectors $|g_1\rangle \otimes |e_2\rangle$ and $|e_1\rangle \otimes |g_2\rangle$ are not more eigenstates of the Hamiltonian of the atomic system. It can be shown that these two vectors states must be substituted by the two entangled states, known in the literature as maximally entangled states [12] [38]

$$|s\rangle = \frac{1}{\sqrt{2}} (|e_1\rangle \otimes |g_2\rangle + |g_1\rangle \otimes |e_2\rangle) \quad (17)$$

and

$$|a\rangle = \frac{1}{\sqrt{2}} (|e_1\rangle \otimes |g_2\rangle - |g_1\rangle \otimes |e_2\rangle) . \quad (18)$$

The Hilbert space of the system is spanned by the collective states of the two identical interacting two-level atoms given respectively by $(|g_1\rangle \otimes |g_2\rangle, |s\rangle, |a\rangle, |e_1\rangle \otimes |e_2\rangle)$. As stated before,

both the states $|s\rangle$ and $|a\rangle$ tell us nothing about the state of the first or second atom separately. Therefore when we include the dipole-dipole interaction between the atoms, we open the possibility of realize entangled states.

In the spin-boson model with the dipole-dipole interaction included, we are considering the question of how does the dipole-dipole interaction change the critical temperature, where again the system exhibit a phase transition from fluorescence to super-radiance. The Hamiltonian for the spin-boson model with the dipole-dipole interaction reads

$$H = \frac{1}{N} \sum_{i \neq j}^N H_{(ij)} \sigma_{(i)}^+ \otimes \sigma_{(j)}^- + I_B \otimes \Omega \sum_{j=1}^N \sigma_{(j)}^z + \sum_{k=1}^{\infty} \omega_k b_k^\dagger b_k \otimes I_Q + \frac{1}{\sqrt{N}} \sum_{j=1}^N \sum_{k=1}^{\infty} g_{ik} (b_k + b_k^\dagger) \otimes (\sigma_{(j)}^+ + \sigma_{(j)}^-). \quad (19)$$

Each two-level atom interact with all other atoms of the ensemble with the same coupling strength, i.e., $H_{(ij)} = \lambda$, and the summation is over all the atoms. This is a model with one infinite range interaction. This model is based in the models describing spin systems with long range interaction [77] [78]. In the next section we discuss first the full Dicke model, showing the presence of a quantum phase transition and also a super-radiant phase at some critical temperature.

3 The functional integral for the fermionic full Dicke model

The typical situation in cavity quantum electrodynamics is the case where a single atom is coupled to a cavity mode. Now suppose N identical two-level atoms in an optical cavity whose linear dimensions are small compared to the wavelength associated to the bosonic mode. This ensemble of identical N two-level atoms interacting linearly with one mode of a bosonic field defines the full Dicke model. The Hamiltonian of the system reads

$$H = I_S \otimes \sum_{j=1}^N \frac{\Omega}{2} \sigma_{(j)}^z + \omega_0 b^\dagger b \otimes I_B + \frac{g}{\sqrt{N}} \sum_{j=1}^N (b + b^\dagger) \otimes (\sigma_{(j)}^+ + \sigma_{(j)}^-). \quad (20)$$

In the above equation g is the coupling constant between the atoms and the single mode of the bosonic field. The b and b^\dagger are the boson annihilation and creation operators of mode excitations that satisfy the usual commutation relation rules.

The aim of this section is to prove that a model with an interaction Hamiltonian where we take into account non-resonant processes in which the two-level systems and the bosonic modes are excited or de-excited simultaneously, presents a phase transition from normal to super-radiant state at some temperature with the presence of a condensate and also a quantum phase transition at some critical coupling. We should mention that the results of this section are not original, but

by completeness we include in the paper. The Euclidean action of the model is S , where H_F is the full Hamiltonian for the fermionic full Dicke model. This quantity can be written as

$$\begin{aligned}
H_F = & \omega_0 b^*(\tau) b(\tau) + \frac{\Omega}{2} \sum_{i=1}^N \left(\alpha_i^*(\tau) \alpha_i(\tau) - \beta_i^*(\tau) \beta_i(\tau) \right) + \\
& + \frac{g_1}{\sqrt{N}} \sum_{i=1}^N \left(\alpha_i^*(\tau) \beta_i(\tau) b(\tau) + \alpha_i(\tau) \beta_i^*(\tau) b^*(\tau) \right) + \\
& + \frac{g_2}{\sqrt{N}} \sum_{i=1}^N \left(\alpha_i(\tau) \beta_i^*(\tau) b(\tau) + \alpha_i^*(\tau) \beta_i(\tau) b^*(\tau) \right). \tag{21}
\end{aligned}$$

Note we are introducing two coupling constants, g_1 and g_2 , for the rotating and counter-rotating terms, respectively. As we discussed before, the main reason for this is that we are interested in to identify the contribution of the real and virtual processes in the phase transition with the formation of the condensate. For this purpose we must to find the partition function of the system, therefore we calculate the formal quotient $\frac{Z}{Z_0}$, being defined by the equation:

$$\frac{Z}{Z_0} = \frac{\int [d\eta] e^S}{\int [d\eta] e^{S_0}}, \tag{22}$$

where the functional integral is with respect to the complex functions $b^*(\tau)$ and $b(\tau)$ and Grassmann Fermi fields $\alpha_i^*(\tau)$, $\alpha_i(\tau)$, $\beta_i^*(\tau)$ and $\beta_i(\tau)$. Here the expression $[d\eta]$ is the functional measure. It is possible to show that an expression for $\frac{Z}{Z_0}$ which is given by [63] [67]

$$\begin{aligned}
\frac{Z}{Z_0} = & \left[\left(1 - a(0) + 2c(0) \right) \left(1 - a(0) - 2c(0) \right) \right]^{-1/2} \\
& \prod_{\omega > 0} \left[\left(1 - a(\omega) \right) \left(1 - a(-\omega) \right) - 4c^2(\omega) \right]^{-1} + \\
& + O(N^{-1}), \tag{23}
\end{aligned}$$

where $a(\omega)$ and $c(\omega)$ in the above equation are given, respectively, by

$$a(\omega) = \left(\frac{g_1^2 (\Omega - i\omega)^{-1} + g_2^2 (\Omega + i\omega)^{-1}}{(\omega_0 - i\omega)} \right) \tanh \left(\frac{\beta \Omega}{4} \right) \tag{24}$$

and

$$c(\omega) = \left(\frac{g_1 g_2 \Omega}{(\omega_0^2 + \omega^2)^{1/2} (\Omega^2 + \omega^2)} \right) \tanh \left(\frac{\beta \Omega}{4} \right). \tag{25}$$

To obtain the thermodynamic limit we have to take the limit ($N \rightarrow \infty$) in Eq. (23). We turn out to the discussion concerning the local elementary excitation of the ground state. To find the energy spectrum of the bosonic collective excitations we have to use the equation

$$4c^2(\omega) - \left(1 - a(\omega)\right) \left(1 - a(-\omega)\right) = 0, \quad (26)$$

and making the analytic continuation ($i\omega \rightarrow E$), we obtain the following equation

$$\begin{aligned} 1 = & - \left[\frac{g_1^4 + g_2^4}{(\omega_0^2 - E^2)(\Omega^2 - E^2)} \right] \tanh^2\left(\frac{\beta\Omega}{4}\right) + \\ & - \left[\frac{g_1^2 g_2^2}{(\omega_0^2 - E^2)} \left(\frac{1}{(\Omega - E)^2} + \frac{1}{(\Omega + E)^2} - \frac{4\Omega^2}{(\Omega^2 - E^2)^2} \right) \right] \tanh^2\left(\frac{\beta\Omega}{4}\right) + \\ & + \left[\frac{g_1^2(\Omega - E)^{-1} + g_2^2(\Omega + E)^{-1}}{(\omega_0 - E)} + \frac{g_1^2(\Omega + E)^{-1} + g_2^2(\Omega - E)^{-1}}{(\omega_0 + E)} \right] \tanh\left(\frac{\beta\Omega}{4}\right). \end{aligned} \quad (27)$$

Solving the above equation for the case $\beta^{-1} = \beta_c^{-1}$ we find the following roots

$$E_1 = 0 \quad (28)$$

and

$$E_2 = \left(\frac{g_1(\Omega + \omega_0)^2 + g_2(\Omega - \omega_0)^2}{(g_1 + g_2)} \right)^{1/2}. \quad (29)$$

Its low energy state of excitation is a Nambu-Goldstone mode, since the continuous $U(1)$ symmetry has been broken. Therefore by using the non-relativistic version for the Goldstone theorem, the energy spectrum can not have a gap above the ground state. Now, let us present the critical temperature and the energy level spectrum of the collective bosonic excitations of the model with the rotating-wave approximation, where $g_1 \neq 0$ and $g_2 = 0$. The result obtained by Popov and Fedotov [79] [80] is recovered, where the equation

$$a(0) = 1 \quad (30)$$

and

$$\frac{g_1^2}{\omega_0\Omega} \tanh\left(\frac{\beta_c\Omega}{4}\right) = 1, \quad (31)$$

gives the inverse of the critical temperature, β_c . It is given by

$$\beta_c = \frac{4}{\Omega} \operatorname{arctanh}\left(\frac{\omega_0\Omega}{g_1^2}\right). \quad (32)$$

The order parameter of the transition is the expectation value of the number of excitation associated to the bosonic mode per atom, i.e., $\lim_{N \rightarrow \infty} \frac{\langle b^\dagger b \rangle}{N} \neq 0$. Note that again ω_0 , Ω and g_1 define also a non-zero critical temperature where the partition function is no more analytic. We may expect a super-radiant phase for the temperature β_c^{-1} given by Eq. (32). The energy level spectrum of the collective Bose excitations in this case is

$$E_1 = 0, \quad (33)$$

and

$$E_2 = \Omega + \omega_0. \quad (34)$$

In this case, there is also a quantum phase transition, i.e., a zero temperature phase transition when the coupling constant g_1 attains the value $g_1 = (\omega_0 \Omega)^{\frac{1}{2}}$. Now we will show that is possible to have a condensate with super-radiance in a system of N two-level atoms coupled with one mode of a Bose field where only virtual processes contribute. In the pure counter-rotating wave case, i.e., $g_1 = 0$ and $g_2 \neq 0$, the inverse of the critical temperature, β_c is given by

$$\beta_c = \frac{4}{\Omega} \operatorname{arctanh} \left(\frac{\omega_0 \Omega}{g_2^2} \right), \quad (35)$$

and the spectrum of the collective Bose excitations given by

$$E_1 = 0, \quad (36)$$

and

$$E_2 = |\Omega - \omega_0|. \quad (37)$$

Again, there is a zero temperature phase transition when $g_2 = (\omega_0 \Omega)^{\frac{1}{2}}$. In this case there is no thermal excitations and the phase transition is driven by the quantum fluctuations. A comment is in order concerning the spectrum of the Bose excitations. In both of the cases: working with the pure counter-rotating or the rotating-wave terms, there is a phase transition. In the case of the rotating-wave approximation $g_1 \neq 0$ and $g_2 = 0$, there is a Nambu-Goldstone mode ($E = 0$). In the pure counter-rotating case $g_1 = 0$ and $g_2 \neq 0$ also there is a Nambu-Goldstone (gapless) mode. We show that it is possible to have a condensate with super-radiance in a system of N two-level atoms coupled with one mode of a Bose field where only virtual processes contribute. Since the energy $E_2 \approx 0$, local elementary excitations of the ground state with low energy can easily be created causing a significant fluctuation effect. Unfortunately we are not able to evaluate these effect in the systems.

4 The functional integral for the spin-boson model with dipole-dipole coupling

The aim of this section is to prove that the introduction of the dipole-dipole interaction does not modify the temperature of the phase transition from normal to super-radiant state and the presence of a condensate. Also, the energy spectrum of the collective bosonic excitations of the model is the same as in the full Dicke model. Introducing the dipole-dipole coupling in the full Dicke model, where an ensemble of identical N atoms interacts linearly with one mode of a bosonic field, in this section we use the same approach of functional integrals to study the thermodynamic of the model [63] [67] [79] [80]. We would like to stress that in this new situation, the Grassmann functional integrals are not Gaussian. In order to perform the integrals we introduce auxiliary variables in the calculation. The total Hamiltonian of this spin-boson model is defined by:

$$H_T = \frac{1}{N} \sum_{i \neq j}^N H_{(ij)} \sigma_{(i)}^+ \otimes \sigma_{(j)}^- + I_B \otimes \Omega \sum_{j=1}^N \sigma_{(j)}^z + \omega_0 b^\dagger b \otimes I_Q + \frac{g}{\sqrt{N}} \sum_{j=1}^N (b + b^\dagger) \otimes (\sigma_{(j)}^+ + \sigma_{(j)}^-). \quad (38)$$

In the above equation the quantity g is the coupling constant between the atoms and the single mode of the bosonic field. The b and b^\dagger are the boson annihilation and creation operators of mode excitations that satisfy the usual commutation relation rules. As usual, we are using the pseudo-spin operators $\sigma_{(j)}^+$, $\sigma_{(j)}^-$ and $\sigma_{(j)}^z$ which satisfy the standard angular momentum commutation relations corresponding to spin $\frac{1}{2}$ operators. We are also shifting the zero of energy $\frac{1}{2}(\omega_1 + \omega_2)$ for each atom and defining $\Omega = \omega_2 - \omega_1$.

Let us define the Fermi raising and lowering operators α_i^\dagger , α_i , β_i^\dagger and β_i , that satisfy the anti-commutator relations $\alpha_i \alpha_j^\dagger + \alpha_j^\dagger \alpha_i = \delta_{ij}$ and $\beta_i \beta_j^\dagger + \beta_j^\dagger \beta_i = \delta_{ij}$. We can also define the following bilinear combination of Fermi operators, $\alpha_i^\dagger \alpha_i - \beta_i^\dagger \beta_i$, $\alpha_i^\dagger \beta_i$ and finally $\beta_i^\dagger \alpha_i$. Note that $\sigma_{(i)}^z$, $\sigma_{(i)}^+$ and $\sigma_{(i)}^-$ obey the same commutation relations as the above bilinear combination of Fermi operators. Therefore, we can change the pseudo-spin operators of this spin-boson model by using the bilinear combination of Grassmann Fermi fields

$$\sigma_{(i)}^z \longrightarrow (\alpha_i^\dagger \alpha_i - \beta_i^\dagger \beta_i), \quad (39)$$

$$\sigma_{(i)}^+ \longrightarrow \alpha_i^\dagger \beta_i, \quad (40)$$

and finally

$$\sigma_{(i)}^- \longrightarrow \beta_i^\dagger \alpha_i. \quad (41)$$

Using the Eq. (38), the Euclidean fermionic action for the model can be written as

$$S = \int_0^\beta d\tau \left(b^*(\tau) \frac{\partial b(\tau)}{\partial \tau} + \sum_{i=1}^N \left(\alpha_i^*(\tau) \frac{\partial \alpha_i(\tau)}{\partial \tau} + \beta_i^*(\tau) \frac{\partial \beta_i(\tau)}{\partial \tau} \right) \right) - \int_0^\beta d\tau H_F(\tau). \quad (42)$$

The Euclidean action S is given by Eq. (42) where H_F is the full Hamiltonian for the fermionic full Dicke model with dipole-dipole interaction. As in the previous section, introducing two coupling constants, g_1 and g_2 , for the rotating and counter-rotating terms, the Hamiltonian becomes

$$\begin{aligned} H_F &= \omega_0 b^*(\tau) b(\tau) + \frac{\Omega}{2} \sum_{i=1}^N \left(\alpha_i^*(\tau) \alpha_i(\tau) - \beta_i^*(\tau) \beta_i(\tau) \right) + \\ &+ \frac{1}{N} \sum_{i \neq j}^N H_{(ij)} \alpha_i^*(\tau) \beta_i(\tau) \beta_j^*(\tau) \alpha_j(\tau) + \frac{g_1}{\sqrt{N}} \sum_{i=1}^N \left(\alpha_i^*(\tau) \beta_i(\tau) b(\tau) + \alpha_i(\tau) \beta_i^*(\tau) b^*(\tau) \right) + \\ &+ \frac{g_2}{\sqrt{N}} \sum_{i=1}^N \left(\alpha_i(\tau) \beta_i^*(\tau) b(\tau) + \alpha_i^*(\tau) \beta_i(\tau) b^*(\tau) \right). \end{aligned} \quad (43)$$

Let us define the formal quotient of two functional integrals, i.e., the partition function of the interacting model and the partition function of the free model. Therefore, we are interested in to calculate the quantity given by Eq. (22) where $S = S(b, b^*, \alpha, \alpha^\dagger, \beta, \beta^\dagger)$ is the Euclidean action given by Eq. (42). Let us define $S_0 = S_0(b, b^*, \alpha, \alpha^\dagger, \beta, \beta^\dagger)$ which is the free Euclidean action for the free single bosonic mode and the free two-level atoms and finally $[d\eta]$ is the path integral measure. Note that in Eq. (22) we have functional integrals with respect to the complex functions $b^*(\tau)$ and $b(\tau)$ and Grassmann Fermi fields $\alpha_i^*(\tau)$, $\alpha_i(\tau)$, $\beta_i^*(\tau)$ and $\beta_i(\tau)$. Since we are using thermal equilibrium boundary conditions in the imaginary time formalism, the integration variables in Eq. (22) obey periodic boundary conditions for the Bose field, i.e., $b(\beta) = b(0)$, and anti-periodic boundary conditions for the Grassmann Fermi fields i.e., $\alpha_i(\beta) = -\alpha_i(0)$ and $\beta_i(\beta) = -\beta_i(0)$. The free action of the bosonic field is given by

$$S_0(b, b^*) = \int_0^\beta d\tau \left(b^*(\tau) \frac{\partial b(\tau)}{\partial \tau} - \omega_0 b^*(\tau) b(\tau) \right). \quad (44)$$

In order to obtain the effective action of the bosonic mode we must integrate over the Grassmann Fermi fields in the expression given by Eq. (22). The problem now confronting us is the fact that in the action given by Eq. (43) there is a non-Gaussian term, corresponding to the dipole-dipole interaction. Although it is not possible to integrate this term directly, using auxiliary variables of integration we can circumvent this difficult. The following integral is helpful

$$\exp \left(-\frac{1}{N} \sum_{i,j=1}^N x_i^* A_{ij} x_j \right) = (\det A)^{-1} \int \prod_{i=1}^N \frac{dy_i^* dy_i}{-2\pi i} e^{\sum_{i,j=1}^N y_i^* A_{ij}^{-1} y_j + \frac{1}{\sqrt{N}} \sum_{i=1}^N y_i x_i^* + \frac{1}{\sqrt{N}} \sum_{i=1}^N y_i^* x_i}, \quad (45)$$

where this integral is valid for commuting variables. To continue we have to study the non-Gaussian contribution to the partition function. In terms of the Grassmann variables we define R as

$$\begin{aligned} R &= \exp \left(-\frac{1}{N} \sum_{i,j=1}^N \int_0^\beta d\tau \alpha_i^*(\tau) \beta_i(\tau) H_{ij} \beta_j^*(\tau) \alpha_j(\tau) \right) \\ &= \exp \left(-\frac{1}{N} \sum_{i,j=1}^N \int_0^\beta d\tau \left(\beta_i^*(\tau) \alpha_i(\tau) \right)^* H_{ij} \beta_j^*(\tau) \alpha_j(\tau) \right), \end{aligned} \quad (46)$$

and since each pair of Grassmann variables $\alpha_i \beta_i^*$ and $\alpha_j \beta_j^*$ commutes, we have $(\alpha_i \beta_i^*)(\alpha_j \beta_j^*) = (\alpha_j \beta_j^*)(\alpha_i \beta_i^*)$. Therefore in the functional version of the model we can substitute the Gaussian integral given in Eq. (45) in Eq. (46). We get

$$R = a_0 (\det H)^{-1} \int [dr] e^{\sum_{i,j=1}^N \int_0^\beta d\tau r_i^*(\tau) H_{ij}^{-1} r_j(\tau) + \frac{1}{\sqrt{N}} \sum_{i=1}^N \int_0^\beta d\tau \left(r_i(\tau) \alpha_i^*(\tau) \beta_i(\tau) + r_i^*(\tau) \beta_i^*(\tau) \alpha_i(\tau) \right)}, \quad (47)$$

where the auxiliary fields $r_i(\tau)$ satisfy periodic boundary conditions, i.e., $r_i(0) = r_i(\beta)$ and also $r_i^*(0) = r_i^*(\beta)$. In the above expression the quantity a_0 is a numerical factor that can be absorbed in the normalization factor and the term $[dr] \equiv \prod_i [dr_i][dr_i^*]$ is the functional measure for the auxiliary complex fields. Substituting the Eq. (47) in the partition function Z , we obtain

$$Z = a_0 (\det H)^{-1} \int [dr] e^{-\sum_{i,j=1}^N \int_0^\beta d\tau r_i^*(\tau) H_{ij}^{-1} r_j(\tau)} \int [d\eta] e^{S_r}, \quad (48)$$

where in Eq. (48) we have $S_r \equiv S_r(r_i, r_i^*, b, b^*, \alpha_i, \dots)$. Consequently the last functional integral $\int [d\eta] e^{S_r}$ depends on the auxiliary fields $r_i^*(\tau)$ and $r_i(\tau)$. Moreover, the action S_r can be separated into a free action for the bosons and a Gaussian fermionic part. The action S_r can be written in the form

$$S_r = S_0(b, b^*) + \int_0^\beta d\tau \sum_{i=1}^N \rho_i^\dagger(\tau) M_r \rho_i(\tau). \quad (49)$$

We would like to stress that in Eq. (49), instead of writing $M(r_i, r_i^*, b^*, b)$, for simplicity we use the notation M_r , i.e., $M_r \equiv M(r_i, r_i^*, b^*, b)$. The column matrix $\rho_i(\tau)$ is given in terms of the Grassmann Fermi fields

$$\begin{aligned} \rho_i(\tau) &= \begin{pmatrix} \beta_i(\tau) \\ \alpha_i(\tau) \end{pmatrix}, \\ \rho_i^\dagger(\tau) &= \begin{pmatrix} \beta_i^*(\tau) & \alpha_i^*(\tau) \end{pmatrix} \end{aligned} \quad (50)$$

and the matrix M_r is given by

$$M_r = \begin{pmatrix} \partial_\tau + \frac{\Omega}{2} & -N^{-1/2} \left(g_1 b^*(\tau) + g_2 b(\tau) - r_i^*(\tau) \right) \\ -N^{-1/2} \left(g_1 b(\tau) + g_2 b^*(\tau) - r_i(\tau) \right) & \partial_\tau - \frac{\Omega}{2} \end{pmatrix}. \quad (51)$$

The complex functions and Grassmann Fermi fields $r_i(\tau)$, $b(\tau)$, $\alpha_i(\tau)$ and $\beta_i(\tau)$ can be represented in terms of a Fourier expansion. Therefore, we have

$$r_i(\tau) = \beta^{-1/2} \sum_v r_i(v) e^{iv\tau}, \quad (52)$$

$$b(\tau) = \beta^{-1/2} \sum_\omega b(\omega) e^{i\omega\tau}, \quad (53)$$

and

$$\rho_i(\tau) = \beta^{-1/2} \sum_p \rho_i(p) e^{ip\tau}. \quad (54)$$

Since the complex functions $r_i(\tau)$ and $b(\tau)$ obey periodic boundary conditions, and the Grassmann Fermi fields $\alpha_i(\tau)$ and $\beta_i(\tau)$ obey anti-periodic boundary conditions, we have that $v = \frac{2\pi n}{\beta}$, $\omega = \frac{2\pi n}{\beta}$ and $p = \frac{(2n+1)\pi}{\beta}$, for integer n . They are respectively the bosonic and fermionic Matsubara frequencies. Substituting the Fourier expansions in the action given by Eq. (49) we get

$$S_r = \sum_\omega (i\omega - \omega_0) b^*(\omega) b(\omega) + \sum_{p,q} \sum_{i=1}^N \rho_i^\dagger(p) M_{pq} \rho_i(q), \quad (55)$$

where the matrix $M_{pq} \equiv M_{pq}(r_i, r_i^*, b, b^*)$ is given by

$$M_{pq} = \begin{pmatrix} (ip + \Omega/2)\delta_{pq} & -(N\beta)^{-1/2} Q_{pq}^\dagger \\ -(N\beta)^{-1/2} Q_{pq} & (ip - \Omega/2)\delta_{pq} \end{pmatrix} \quad (56)$$

and the function $Q_{pq} \equiv Q_{pq}(r_i, b, b^*)$ can be written as

$$Q_{pq} = g_1 b(p - q) + g_2 b^*(q - p) - r_i(p - q). \quad (57)$$

Since the integrals with respect to the Fermi fields are Gaussian, we may integrate over these Grassmann variables. This procedure yields

$$\int [d\eta(\rho)] \exp \left(\sum_{p,q} \sum_{i=1}^N \rho_i^\dagger(p) M_{pq} \rho_i(q) \right) = \prod_{i=1}^N \det M_i, \quad (58)$$

where the matrix $M_i \equiv M_i(r_i, r_i^*, b^*, b)$ is a block matrix of the following form

$$M_i = \begin{pmatrix} iP + \frac{\Omega}{2} I & -(N\beta)^{-1/2} Q^\dagger \\ -(N\beta)^{-1/2} Q & iP - \frac{\Omega}{2} I \end{pmatrix}. \quad (59)$$

In the above equation I is the identity matrix and the components of matrix P are $P_{pq} = p \delta_{pq}$ and Q was defined in Eq. (57). Our aim is to present the quotient $\frac{Z}{Z_0}$, defined in Eq. (22). Combining the results obtained in Eq. (48), Eq. (49) and Eq. (58) we get that $\frac{Z}{Z_0}$ is given by

$$\frac{(\det H)^{-1} \int [dr] e^{\sum_{i,j=1}^N \sum_v r_i^*(v) H_{ij}^{-1} r_j(v)} \int [d\eta(b)] e^{\sum_\omega (i\omega - \omega_0) b^*(\omega) b(\omega)} \prod_{j=1}^N \det M_j}{\int [d\eta(b)] e^{\sum_\omega (i\omega - \omega_0) b^*(\omega) b(\omega)} \det^N M(0,0)} \quad (60)$$

where the functional measure $[d\eta(b)]$ is defined by

$$[d\eta(b)] = \prod_\omega db(\omega) db^*(\omega). \quad (61)$$

In the Eq. (60) the matrix $M(0,0)$ is given by

$$M(0,0) = \begin{pmatrix} iP + \frac{\Omega}{2} I & 0 \\ 0 & iP - \frac{\Omega}{2} I \end{pmatrix}. \quad (62)$$

In order to simplify the calculations, let us change variables in the following way:

$$b(\omega) \rightarrow \left(\frac{\pi}{(\omega_0 - i\omega)} \right)^{1/2} b(\omega) \quad (63)$$

and

$$b^*(\omega) \rightarrow \left(\frac{\pi}{(\omega_0 - i\omega)} \right)^{1/2} b^*(\omega). \quad (64)$$

Note that Eq. (64) is not the complex conjugate of Eq. (63). It is not difficult to see that after these changes of variables, the denominator of the Eq. (60) turns out to be equal to unity

$$\int [d\eta(b)] \exp \left(-\pi \sum_\omega b^*(\omega) b(\omega) \right) = 1. \quad (65)$$

We can express the ratio $\frac{Z}{Z_0}$ by the integral

$$\frac{Z}{Z_0} = (\det H)^{-1} \int [dr] e^{\sum_{i,j=1}^N \sum_v r_i^*(v) H_{ij}^{-1} r_j(v)} \int [d\eta(b)] \exp \left(S_{eff}(b) \right), \quad (66)$$

where after doing the Fourier transform for the auxiliary fields the functional measure becomes $[dr] = \prod_i^N \prod_\omega dr_i(\omega) dr_i^*(\omega)$. The effective action of the bosonic mode, $S_{eff}(b)$ is given by

$$S_{eff} = -\pi \sum_\omega b^*(\omega) b(\omega) + \sum_{i=1}^N \ln \det(I + A_i). \quad (67)$$

The matrix A_i in the determinant of the above equation is given by

$$\det(I + A_i) = \det\left(M^{-1/2}(0, 0) M_i(b^*, b) M^{-1/2}(0, 0)\right). \quad (68)$$

Performing the product of matrices we obtain the matrix A afterwards. Therefore we can write

$$A_i = \begin{pmatrix} 0 & B_i \\ -C_i & 0 \end{pmatrix}. \quad (69)$$

The components of the matrices B_i and C_i are given by

$$\begin{aligned} (B_i)_{pq} &= -\left(\frac{1}{\beta N}\right)^{\frac{1}{2}} \left(iq - \frac{\Omega}{2}\right)^{-\frac{1}{2}} \left(ip + \frac{\Omega}{2}\right)^{-\frac{1}{2}} \times \\ &\times \left(\frac{\sqrt{\pi} g_1 b^*(q-p)}{\sqrt{\omega_0 - i(q-p)}} + \frac{\sqrt{\pi} g_2 b(p-q)}{\sqrt{\omega_0 - i(p-q)}} - r_i^*(q-p)\right) \end{aligned} \quad (70)$$

and

$$\begin{aligned} (C_i)_{pq} &= \left(\frac{1}{\beta N}\right)^{\frac{1}{2}} \left(ip - \frac{\Omega}{2}\right)^{-\frac{1}{2}} \left(iq + \frac{\Omega}{2}\right)^{-\frac{1}{2}} \times \\ &\times \left(\frac{\sqrt{\pi} g_1 b(p-q)}{\sqrt{\omega_0 - i(p-q)}} + \frac{\sqrt{\pi} g_2 b^*(q-p)}{\sqrt{\omega_0 - i(q-p)}} - r_i(p-q)\right). \end{aligned} \quad (71)$$

In order to perform the functional integral given by Eq. (66) we must to find a manageable expression for $\det(I + A)$. For this case we can use the following identity

$$\det(I + A_i) = \det(I + B_i C_i) \rightarrow \exp\left(\text{tr}(B_i C_i)\right). \quad (72)$$

So using this approximation given by Eq. (72) we can find an expression for the ratio $\frac{Z}{Z_0}$ defined in Eq. (66) of the following form

$$\frac{Z}{Z_0} = (\det H)^{-1} \int [dr] e^{\sum_{i,j=1}^N \sum_v r_i(v) \left(H_{ij}^{-1} - \frac{1}{N} \tanh\left(\frac{\beta\Omega}{4}\right) \frac{\delta_{ij}}{iv-\Omega}\right) r_j^*(v)} \frac{Z_r}{Z_0}, \quad (73)$$

where

$$\begin{aligned} \frac{Z_r}{Z_0} = & \int [d\eta(b)] e^{-\pi \sum_{\omega} b^*(\omega) (1-a(\omega)) b(\omega) + \pi \sum_{\omega} (b(\omega) c(\omega) b(-\omega) + b^*(\omega) c(\omega) b^*(-\omega))} \times \\ & \times e^{\frac{\pi}{N} \sum_{\omega} (d_1(\omega) b(\omega) + d_2(\omega) b^*(\omega))}. \end{aligned} \quad (74)$$

In the Eq. (73) the coefficients $a(\omega)$ and $c(\omega)$ are given respectively by

$$a(\omega) = \left(\frac{g_1^2 (\Omega - i\omega)^{-1} + g_2^2 (\Omega + i\omega)^{-1}}{(\omega_0 - i\omega)} \right) \tanh \left(\frac{\beta \Omega}{4} \right), \quad (75)$$

$$c(\omega) = \left(\frac{g_1 g_2 \Omega}{(\omega_0^2 + \omega^2)^{1/2} (\Omega^2 + \omega^2)} \right) \tanh \left(\frac{\beta \Omega}{4} \right), \quad (76)$$

$$d_1(\omega) = -\frac{1}{\sqrt{\pi} \sqrt{\omega_0 - i\omega}} \left(\frac{g_1}{\Omega - i\omega} \sum_i r_i^*(\omega) + \frac{g_2}{\Omega + i\omega} \sum_i r_i(-\omega) \right) \tanh \left(\frac{\beta \Omega}{4} \right) \quad (77)$$

and

$$d_2(\omega) = -\frac{1}{\sqrt{\pi} \sqrt{\omega_0 - i\omega}} \left(\frac{g_1}{\Omega - i\omega} \sum_i r_i(\omega) + \frac{g_2}{\Omega + i\omega} \sum_i r_i^*(-\omega) \right) \tanh \left(\frac{\beta \Omega}{4} \right). \quad (78)$$

Performing the Gaussian integral in the bosonic variables in the Eq. (74), we finally obtain

$$\begin{aligned} \frac{Z_r}{Z_0} = & I_0 \times \exp \left(\frac{1}{N^2} \sum_{\omega} \frac{A_1(\omega)}{(1-a(-\omega))(1-a(\omega)) - 4c^2(\omega)} \sum_{ij} r_i(\omega) r_j^*(\omega) + \right. \\ & \left. + \sum_{\omega} \frac{A_2(\omega)}{(1-a(-\omega))(1-a(\omega)) - 4c^2(\omega)} \left(\sum_{ij} r_i(\omega) r_j(-\omega) + \sum_{ij} r_i^*(\omega) r_j^*(-\omega) \right) \right), \end{aligned} \quad (79)$$

where I_0 , $A_1(\omega)$ and $A_2(\omega)$ are defined respectively by

$$\begin{aligned} I_0 = & \frac{1}{(1-a(0) - 2c(0))^{1/2} (1-a(0) + 2c(0))^{1/2}} \times \\ & \times \prod_{\omega} \frac{1}{(1-a(-\omega))(1-a(\omega)) - 4c^2(\omega)}, \\ A_1(\omega) = & \frac{\tanh \left(\frac{\beta \Omega}{4} \right)}{\pi (\Omega - i\omega)^2} \left(\frac{g_1^2}{\omega_0 - i\omega} + \frac{g_2^2}{\omega_0 + i\omega} - \tanh \left(\frac{\beta \Omega}{4} \right) \frac{(g_1^2 - g_2^2)^2}{(\omega_0^2 - \omega^2)(\Omega + i\omega)} \right), \\ A_2(\omega) = & \frac{g_1 g_2 \omega_0}{\pi (\omega_0^2 - \omega^2)(\Omega^2 + \omega^2)} \tanh \left(\frac{\beta \Omega}{4} \right). \end{aligned} \quad (80)$$

Comparing the partition function for the full Dicke model in Eq. (23), with the partition function of the model with dipole-dipole coupling in Eqs. (73) and (79), we can see that the poles in the partition function are the same and therefore the temperature of the phase transition from normal to super-radiant state and the presence of a condensate is not affected by the dipole-dipole coupling. The energy level spectrum of the collective bosonic excitations of the model is the same as the original full Dicke model. The quantum phase transition is not modified in this situation. An important question here, is why the full Dicke model with the dipole-dipole interaction has the same critical behavior as the original model?

5 Conclusions

With the development of quantum information and its application in computation and communication the entangled states have been attracted enormous interest, since several quantum protocols can be realized exclusively with the help of entangled states. For example new features in cryptography becomes possible utilizing the non-classical properties of entangled pairs of particles. The basic problem that arises in this area of research is how to measuring entanglement in many-body systems. The Schmidt decomposition can be used to measure correlation between two sub-systems in a joint pure state. An important property is that the reduced matrices (see appendix A) of both sub-systems written in the Schmidt basis are diagonal and have the same positive spectrum. For more than two entangled sub-systems the Schmidt decomposition is in general impossible to be implemented. For mixed states of two or more subsystems there is not a Schmidt decomposition, and different measures of entanglement are nonequivalent. The structure of entanglement in many-body systems is much more difficult to study then for bipartite systems.

In a system of N atoms, the presence of the dipole-dipole interaction generates entangled states. If entangled has been created in a portion of the many-body system, the critical behavior of the system can changes revealing the strong correlation between the atoms. In this work we investigate if the entanglement between the atoms, generated by the dipole-dipole interaction, changes the critical properties of the system, as the critical temperature that characterize the phase transition from fluorescent to super-radiant phase and also the spectrum of the bosonic excitation of the model.

First we consider the full Dicke model composed by a single bosonic mode and an ensemble of N identical two-level atoms. Assuming that the system is in thermal equilibrium with a reservoir at temperature β^{-1} , we consider the situation where the coupling between the bosonic mode and the atoms generates resonant and non-resonant processes. Secondly we consider the full Dicke spin-boson model with the dipole-dipole interaction between the atoms. We show that, introducing the dipole-dipole interaction term, the critical temperature that characterize the phase transition from fluorescent to super-radiant phase does not change. Also, the spectrum of the bosonic excitations of the model is unaffected by the dipole-dipole interaction. Finally, the quantum phase transition

is not modified by the dipole-dipole interaction. Furthermore, as discussed previously there is a zero temperature phase transition when $g_1 = 0$ and $g_2 \neq 0$. In this case there are no thermal excitations and the phase transition is driven by the quantum fluctuations. An important point is that there is a close connection between quantum phase transition and entanglement. A quantum phase transition can be characterized by long range correlations. As the same way, the definition of entangled pure state is that in the system at least one pair of the observables has quantum correlations.

We might ask why we have obtained this result, since we expect at principle that the non-Gaussian terms in the Hamiltonian can modify phase transition properties. Remember that the spin-boson Hamiltonian is a particular case of the Hamiltonian describing a system of bound charges and a radiation field in the dipole approximation [81], where we are assuming that the system is confined in a region whose linear dimensions are small compared with the wavelength of the bosonic mode. In the full Dicke model, field non-uniformity is disregarded. All atoms are in the same environment. The introduction of the dipole-dipole interaction with infinite range does not modify this situation. Each two-level atoms is coupled to infinitely many others. The above discussion leads us to conclude that we are solving mean-field models. Other heuristic argument is related to the fact that mean field theories give exactly the right description of critical properties for large d , where d is the dimensionality of space. Since our results are independent of the dimensionality, we are solving mean field models. That is the reason why the dipole-dipole interaction is not able to modify the critical properties for this cooperative phenomena of super-radiance.

We conjecture that the critical properties are different if we go beyond the mean field approximation. First let us assume that linear dimensions of the system are not small compared with the wavelength associated to the bosonic mode. In this case we have to take into account the spatial variation in the coupling coefficient between the pseudo-spin operators $\sigma_{(j)}^+$, $\sigma_{(j)}^-$ and the boson annihilation and creation operators of mode excitations. We note that Li and co-workers [82] investigated the quantum phase transition and the super-radiant phase in the full Dicke model assuming that the dimensions of the atomic ensemble are much larger than the wavelength of the bosonic mode. This system has a critical temperature of the phase transition from normal to super-radiant state and also the presence of a condensate. With the introduction of the dipole-dipole term, the critical temperature of the super-radiant phase transition must change. This subject is under investigation by the authors.

6 Acknowledgements

N. F. Svaiter would like to acknowledge the hospitality of the Departamento de Física, Universidade Federal da Paraíba, where part of this paper was carried out. This paper was supported in part by Conselho Nacional de Desenvolvimento Científico e Tecnológico do Brazil (CNPq).

A The theory of pure and mixed states and the reduced density operator.

In the theory of quantum systems, the notion of composite quantum systems is fundamental. The Hilbert space of a composite quantum system is the tensor product space of the Hilbert spaces describing its sub-systems. Being more specific, let us consider a bi-partite quantum system S made of two quantum systems $S^{(1)}$ and $S^{(2)}$, with respective Hilbert spaces $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$. The state space \mathcal{H} of the combined system ($S^{(1)} + S^{(2)}$) is given by the tensor product of the Hilbert spaces pertaining to the sub-systems $S^{(1)}$ and $S^{(2)}$. Therefore we have that the total Hilbert space can be written as

$$\mathcal{H} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}. \quad (\text{A.1})$$

If we take fixed orthonormal basis $\{|\varphi_i^{(1)}\rangle\}$ and $\{|\varphi_j^{(2)}\rangle\}$ in $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$ respectively, a general state $|\psi\rangle$ in the tensor product space \mathcal{H} may be written as

$$|\psi\rangle = \sum_{i,j} \alpha_{ij} |\varphi_i^{(1)}\rangle \otimes |\varphi_j^{(2)}\rangle. \quad (\text{A.2})$$

where $\{|\varphi_i^{(1)}\rangle \otimes |\varphi_j^{(2)}\rangle\}$ is a basis of the tensor product space.

Suppose that we have an operator $A^{(1)}$ acting only in $\mathcal{H}^{(1)}$ and an operator $A^{(2)}$ acting only in $\mathcal{H}^{(2)}$. Therefore we define the tensor product $A^{(1)} \otimes A^{(2)}$ showing how does this quantity acts on any element of the basis of the tensor product space. We have

$$\left(A^{(1)} \otimes A^{(2)}\right) \left(|\varphi_i^{(1)}\rangle \otimes |\varphi_j^{(2)}\rangle\right) = \left(A^{(1)} |\varphi_i^{(1)}\rangle\right) \otimes \left(A^{(2)} |\varphi_j^{(2)}\rangle\right). \quad (\text{A.3})$$

This important definition lead us to its linear extension for an arbitrary state $|\psi\rangle$. Using the Eq. (A.2) and the Eq. (A.3) we get

$$\left(A^{(1)} \otimes A^{(2)}\right) |\psi\rangle = \sum_{i,j} \alpha_{ij} \left(A^{(1)} |\varphi_i^{(1)}\rangle\right) \otimes \left(A^{(2)} |\varphi_j^{(2)}\rangle\right). \quad (\text{A.4})$$

We also have that any product of operators acting in \mathcal{H} can be represented as a linear combination of tensor products. We have

$$A = \sum_{\alpha} \left(A_{\alpha}^{(1)} \otimes A_{\alpha}^{(2)}\right). \quad (\text{A.5})$$

The observable of the sub-system $S^{(1)}$ take the form $A^{(1)} \otimes I^{(2)}$, while observable of sub-system $S^{(2)}$ take the form $I^{(1)} \otimes A^{(2)}$.

Suppose that we are interested in the behavior of the sub-system $S^{(1)}$. This behavior is determined by the expectation values of the operators $\{X_{\alpha}^{(1)}\}$, where the label (1) means that each

operator acts on the states of the subsystem $S^{(1)}$ alone, and the index $\alpha = 1, 2, \dots, N$ means operators in $\mathcal{H}^{(1)}$. In other words, the label α denotes the family of commuting operators $([X_\alpha^{(1)}, X_\beta^{(1)}] = 0)$, acting in the state $|\psi\rangle$. Since the state vector $|\psi\rangle$ of the combined system can be expanded using the orthonormal basis $\{|\varphi_i^{(1)}\rangle\}$ and $\{|\varphi_j^{(2)}\rangle\}$ we get for any operator $\{X^{(1)}\}$, its expectation value given by

$$\langle X^{(1)} \rangle = \langle \psi | X^{(1)} | \psi \rangle. \quad (\text{A.6})$$

Note that we drop the indices α in the operator. Using the Eq. (A.2), the fact that the operator $X^{(1)}$ does not act on the vectors that span the Hilbert space $\mathcal{H}^{(2)}$, and the orthonormality of the $\{|\varphi_j^{(2)}\rangle\}$ basis, we get

$$\langle X^{(1)} \rangle = \sum_{ijk} \alpha_{kj}^* \alpha_{ij} \langle \varphi_k^{(1)} | X^{(1)} | \varphi_i^{(1)} \rangle. \quad (\text{A.7})$$

Defining $c_{ik} = \sum_j \alpha_{ij} \alpha_{kj}^*$, we have that the expectation value of the operator $X^{(1)}$ in a general state $|\psi\rangle$, defined by $\langle X^{(1)} \rangle$, is given by

$$\langle X^{(1)} \rangle = \sum_{ik} c_{ik} \langle \varphi_k^{(1)} | X^{(1)} | \varphi_i^{(1)} \rangle, \quad (\text{A.8})$$

or

$$\langle X^{(1)} \rangle = \text{tr} [X^{(1)} \rho^{(1)}], \quad (\text{A.9})$$

where

$$\rho^{(1)} = \sum_{ik} c_{ik} |\varphi_i^{(1)}\rangle \langle \varphi_k^{(1)}|. \quad (\text{A.10})$$

The quantity $\rho^{(1)}$ is called the density operator of the sub-system $S^{(1)}$. The density operator describes the state of the system interacting with other system in the same way as the state vector describes the state of an isolated system.

Is a fundamental interest to measuring entangled in many body systems, where we have usually mixed states. For simplicity, let us consider a two-component quantum system $S^{(1)}$ and $S^{(2)}$ in a pure state $|\Phi\rangle$. A pure state is separable if and only if the reduced density operators represent pure states. Essentially separable states satisfy the classical separability principle. In a bipartite quantum system with an entangled state $|\Phi_s\rangle$, measurements in its sub-systems show that $|\Phi_s\rangle$ contain information in the measurements in $S^{(1)}$ and $S^{(2)}$ and also correlation between the measurements. To test whether the sub-systems $S^{(1)}$ and $S^{(2)}$ are entangled or not, we perform the Schmidt decomposition of the state vector $|\Phi\rangle$. If the Schmidt number is greater than one, the state $|\Phi\rangle$ is an entangled state. Therefore a pure state is entangled if and only if the reduced density operators for the sub-systems describe mixed states. Using the Schmidt decomposition it is possible to evaluate the density operators $\rho_{mixed}^{(1)} = \text{Tr}_{(2)} |\Phi\rangle \langle \Phi|$ and $\rho_{mixed}^{(2)} = \text{Tr}_{(1)} |\Phi\rangle \langle \Phi|$. The two reduced density operators have the same eigenvalues, and the two sub-systems have identical

von Neumann entropies. On the other hand, if the Schmidt number is one, the reduced density operators for $S^{(1)}$ or $S^{(2)}$ represent pure states and the bipartite system is not an entangled state. The von Neumann entropy of a pure state is zero. Suppose the operators $X^{(1)}$ and $X^{(2)}$ defined as $X^{(1)} \otimes I^{(2)}$ and $I^{(1)} \otimes X^{(2)}$. The fluctuation operators are defined as $\delta X^{(1)} = X^{(1)} - \langle X^{(1)} \rangle$ and $\delta X^{(2)} = X^{(2)} - \langle X^{(2)} \rangle$. In practice, a pure state is separable if and only if the quantum fluctuations of all observable (linear self-adjoint operators acting in the respective Hilbert space of each sub-system) are uncorrelated. If at least one pair of linear self-adjoint operators have correlated quantum fluctuations, the pure state is entangled.

Note that the von Neumann entropy attains the maximum $S(\rho) = \ln N$, in the case of completely mixed state where $\rho = \frac{1}{N} \sum_i |\psi_i\rangle \langle \psi_i|$, where N is the dimension of the corresponding Hilbert space. Thus quantity is a monotonically function of the degree of entangled between a pair of sub-systems. Therefore the von Neumann entropy of the density matrix for $S^{(1)}$ or $S^{(2)}$ of the bipartite system is a measurement of entangled [42].

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